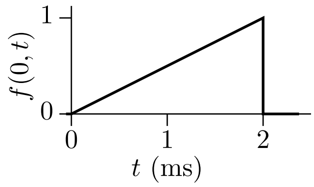
Quantitative Physiology: Cells and Tissues
2.791J/2.794J/6.021J/6.521J/BE.370J/BE.470J/HST.541J
Recitation 11 :: October 24, 2006

Exercise ex2.2.10. Let the function $f(z,t)$ represent a solution to the wave equation. This solution is shown in the following figure as a function of time t at the position $z = 0$.

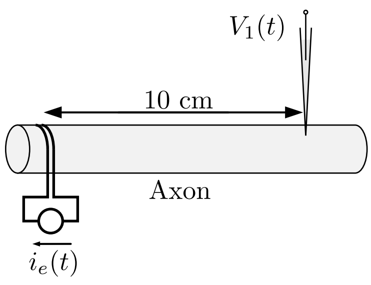


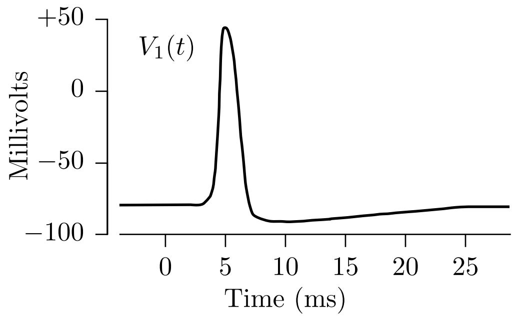
Notice that $f(0,t)$ is non-zero for $0 < t < 2$ ms and zero elsewhere.

a) Suppose that the wave is propagating in the $+z$-direction with a propagation velocity of 100 mm/ms. Plot $f(z,t)$ versus z at time $t = 2$ ms.

b) Suppose that the wave is propagating in the $-z$-direction at a propagation velocity of 100 mm/ms. Plot $f(z,t)$ versus z at time $t = 2$ ms.

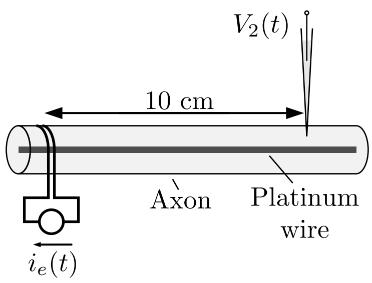
Problem pr2.2.3. A squid axon (500 μm in diameter) is placed in sea water and stimulated electrically at $t = 0$ to produce an action potential, $V_1(t)$, that is recorded at a site 10 cm from the point of stimulation as shown in the following figures.





The resistivity of the axoplasm of this axon is (remarkably enough) 10π Ω-cm. The resistance of the external solution can be assumed to be negligibly small.

A fine platinum wire with a resistance per unit length of 160 Ω/cm is inserted down the entire length of the axon as shown below.



The wire takes up negligible space. The axon is stimulated electrically in an identical manner to produce an action potential $V_2(t)$.

a) Find the conduction velocity, ν_1, of the peak of the action potential before the platinum wire has been inserted.

b) Find the conduction velocity, ν_2, of the peak of the action potential after the platinum wire has been inserted.

c) Sketch $V_2(t)$ on the same time axis as $V_1(t)$.

d) Write an expression for $V_2(t)$ in terms of $V_1(t)$.

SOLUTIONS

Example 2.2.10
Since the wave propagates at constant velocity, it has the form $f(z,t) = g(t - z/\nu)$, where z is in mm, t is in ms, and ν is in mm/ms. Substituting $z = 0$ shows that $f(0,t) = g(t)$, which is the function provided in the problem statement.

a. To find the new plot, notice that $f(z,2) = g(2 - z/\nu)$. Furthermore, $f(z,2) = g(2 - z/100)$ since the velocity $\nu = 100$. To plot $f(z,2)$, we just substitute $z = 0$ in $f(z,2)$ which leads to $f(0,2) = g(2)$ which is given in the figure as 1. Similarly $f(200,2) = g(2 - 200/100) = g(0)$ which is given in the figure as 0. These two critical points determine the whole function, which is plotted in the left panel of Figure 1.

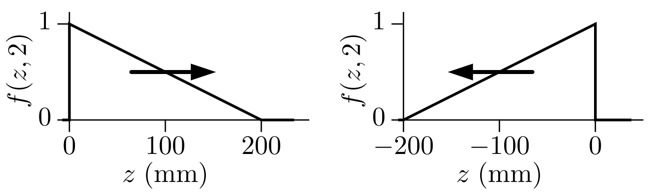


Figure 1: Wave propagation in the $+$ and $-$ z direction. The arrows show the direction of propagation.

b. Following the same reasoning as in part a, $f(z,2) = g(2 + z/100)$ since the velocity $\nu = -100$. To plot $f(z,2)$, we just substitute $z = 0$ in $f(z,2)$ which leads to $f(0,2) = g(2)$ which is given in the figure as 1. Similarly $f(-200,2) = g(2 - 200/100) = g(0)$ which is given in the figure as 0. These two critical points determine the whole function, which is plotted in the right panel of Figure 1.

Problem 2.2.3

a) The conduction velocity of the peak of the action potential is

$$\nu = \frac{10 \text{ cm}}{5 \text{ ms}} = 2 \text{ cm/ms} \ = 20 \text{ m/s}.$$

b) The platinum wire provides another path for longitudinal current flow that is in parallel with the cytoplasm. The resistance per unit length of the platinum wire is $r_{pl} = 160$ Ω/cm. The resistance per unit length of the cytoplasm is $r_{cy} = 10\pi/(\pi)(0.025)^2 = 1.6 \times 10^4$ Ω/cm.

Hence, the parallel combination of these two resistance is $r_i = r_{pl}r_{cy}/(r_{pl} + r_{cy}) \approx r_{pl}$. If the extracellular resistance is negligible, the conduction velocities of the axon are

$$\nu_1 = \sqrt{\frac{\mathcal{K}_m}{2\pi a r_1}} \text{ and } \nu_2 = \sqrt{\frac{\mathcal{K}_m}{2\pi a r_2}},$$

for different values of the intracellular resistance per unit length with all other factors unchanged. Thus,the ratio of conduction velocities is

$$\frac{\nu_1}{\nu_2} = \sqrt{\frac{r_2}{r_1}}.$$

Therefore, $\nu_2 = 20\sqrt{16000/160} = 200$ m/s.

c) With the platinum wire in place, the new conduction velocity is 20000 cm/s. Since the action potential travels 10 cm, the peak is delayed by 0.5 ms as is shown in Figure 2.

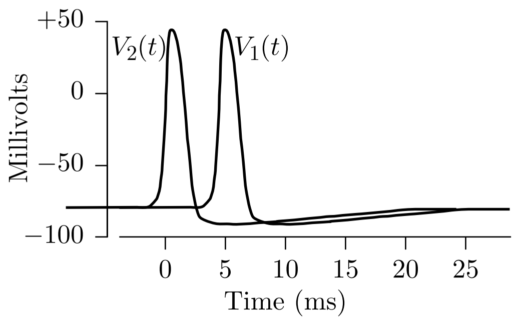


Figure 2: Action potential with ($V_2(t)$) and without ($V_1(t)$) the platinum wire.

d) Each of the two voltages must have the form of a travelling wave. Therefore,

$$V_1(t) = f\left(t - \frac{z}{\nu_1}\right) \text{ and } V_2(t) = f\left(t - \frac{z}{\nu_2}\right),$$

which implies that

$$V_1\left(t + \frac{z}{\nu_1}\right) = f(t) \text{ and } V_2\left(t + \frac{z}{\nu_2}\right) = f(t).$$

Equating these two expressions yields

$$V_1\left(t + \frac{z}{\nu_1}\right) = V_2\left(t + \frac{z}{\nu_2}\right)$$

which implies that

$$V_2(t) = V_1\left(t + \frac{z}{\nu_1} - \frac{z}{\nu_2}\right),$$

so that

$$V_2(t) = V_1\left(t + \frac{10}{2000} - \frac{10}{20000}\right) = V_1\left(t + 4.5 \times 10^{-3}\right)$$

where t in s.