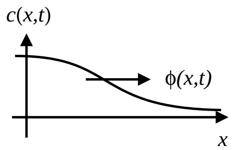
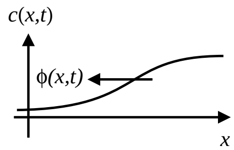
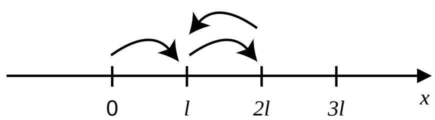
Review of Lecture 2

- Diffusion = transport of solute due to gradient of solute concentration
- Fick's first law: $\phi(x,t) = -D \dfrac{\partial c(x,t)}{\partial x}$
  – inspired by experimental work of Graham
  – developed by analogy to laws for heat and charge transport

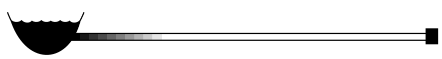




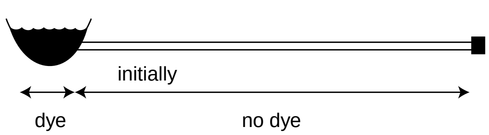
- Microscopic basis: Einstein's random walk model
  – predicts Brownian motion at a microscopic scale
  – predicts Fick's first law at a macroscopic scale

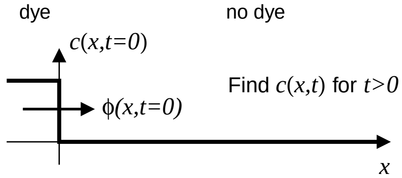


- Demonstration



Apply Fick' s law to dye demonstration



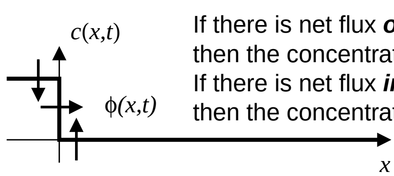


Find $c(x,t)$ for $t>0$

Fick' s law:

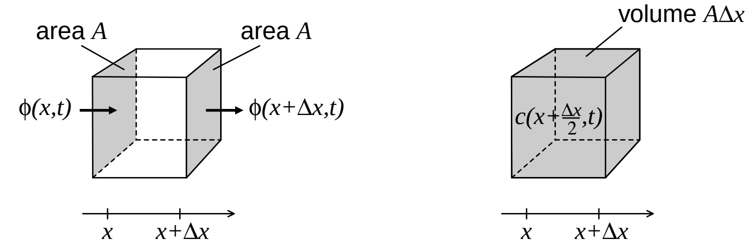
$$\phi(x,t) = -D \frac{\partial c(x,t)}{\partial x}$$

- provides information about time "t" only
- need new information to get from time "t" to time "$t+\Delta t$"



If there is net flux ***out*** of a region,
then the concentration in that region must ***fall***.
If there is net flux ***into*** a region,
then the concentration in that region must ***rise***.
→ conservation of solute

Continuity Equation



Amount of solute entering through edges during $(t,t+\Delta t)$

Change in amount of solute in volume from t to $t+\Delta t$

—— left —— —— right —— —— time 2 —— —— time 1 ——

$$\phi(x,t+\tfrac{\Delta t}{2})\,A\,\Delta t - \phi(x+\Delta x,t+\tfrac{\Delta t}{2})\,A\,\Delta t = c(x+\tfrac{\Delta x}{2},t+\Delta t)\,A\,\Delta x - c(x+\tfrac{\Delta x}{2},t)\,A\,\Delta x$$

↑ equal if solute is neither created nor destroyed

$$\frac{\phi(x,t+\frac{\Delta t}{2}) - \phi(x+\Delta x,t+\frac{\Delta t}{2})}{\Delta x} = \frac{c(x+\frac{\Delta x}{2},t+\Delta t) - c(x+\frac{\Delta x}{2},t)}{\Delta t}$$

Take limit as $\Delta x \to 0$ and $\Delta t \to 0$

$$-\frac{\partial \phi(x,t)}{\partial x} = \frac{\partial c(x,t)}{\partial t}$$

Fick' s first law: $$\phi(x,t) = -D\frac{\partial c(x,t)}{\partial x}$$

Continuity equation: $$-\frac{\partial \phi(x,t)}{\partial x} = \frac{\partial c(x,t)}{\partial t}$$

Diffusion equation: $$\frac{\partial c(x,t)}{\partial t} = D\frac{\partial^2 c(x,t)}{\partial x^2}$$

Time invariant solutions to diffusion equation:
what happens when diffusion procedes for a long time?

$$\frac{\partial \phi(x,t)}{\partial t} \to 0 \qquad \frac{\partial c(x,t)}{\partial t} \to 0 \quad \Rightarrow \quad \frac{\partial \phi(x,t)}{\partial x} \to 0$$

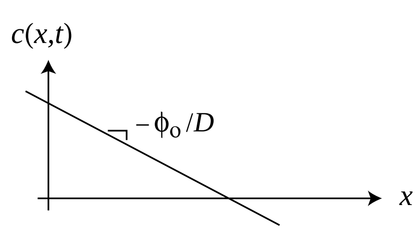
$$\phi(x,t) \to \text{constant}$$

$$\Downarrow$$

$$-D\frac{\partial c(x,t)}{\partial x} = \text{constant} = \phi_o$$

$$\Downarrow$$

$$c(x,t) = \text{linear function of } x = c(x,t_o)$$



If no external sources and sinks $\longrightarrow$ closed system
then sustained fluxes are not possible (require energy)
$\Longrightarrow$ Equilibrium: persistent state of zero flux

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\partial c(x,t)}{\partial t} = 0 \qquad \phi(x,t) = 0$$

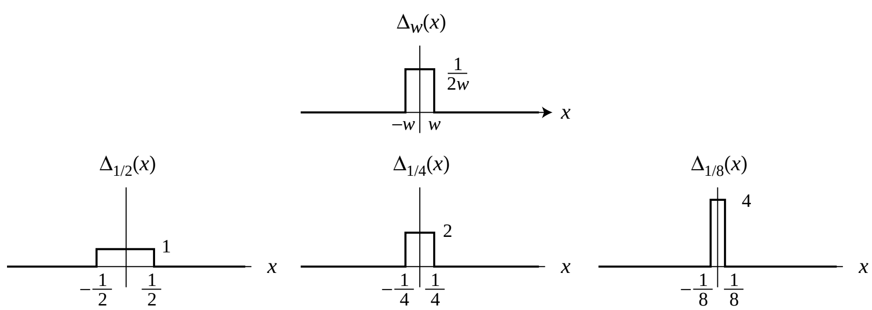
$$\frac{\partial \phi(x,t)}{\partial t} = 0$$

$$\phi(x,t) = -D \frac{\partial c(x,t)}{\partial x} = 0$$

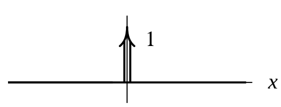
$D = 0$
(not diffusible)

$c(x,t) = c_o$
(constant concentration)

Dirac delta function $\delta(x)$



$$\delta(x) = \lim_{w \to 0} \Delta_w(x)$$



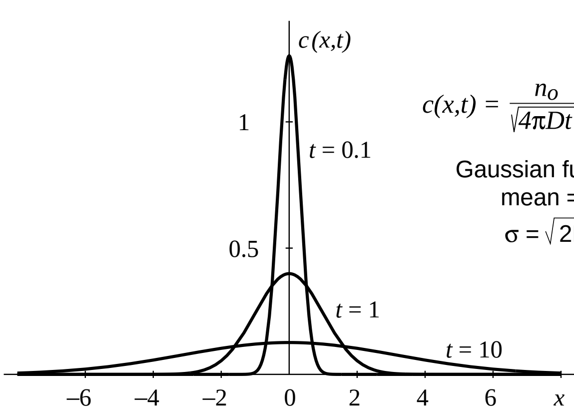
$$\delta(x) = 0 \text{ except at } x=0$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

Impulse Response

Given $c(x,t) = n_o\, \delta(x)$ at $t=0$

Solve $\dfrac{\partial c(x,t)}{\partial t} = D \dfrac{\partial^2 c(x,t)}{\partial x^2}$ for $t>0$



$$c(x,t) = \frac{n_o}{\sqrt{4\pi D t}}\, e^{-x^2/(4Dt)}$$

Gaussian function
mean = 0
$\sigma = \sqrt{2Dt}$

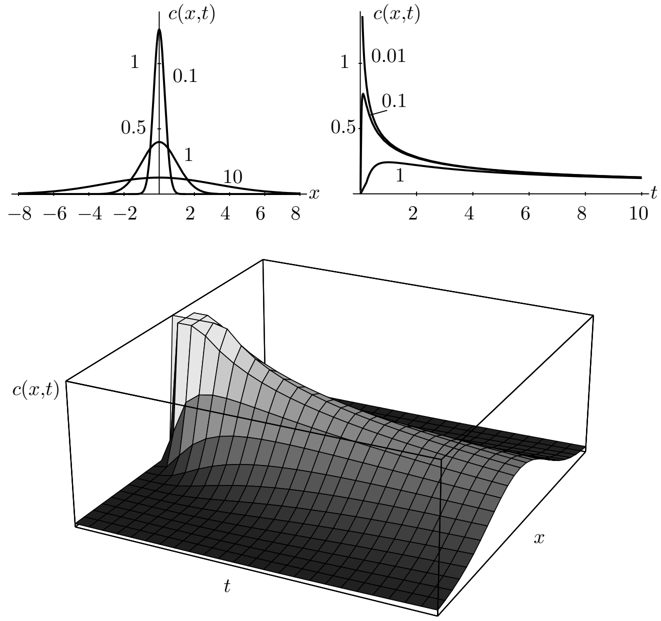
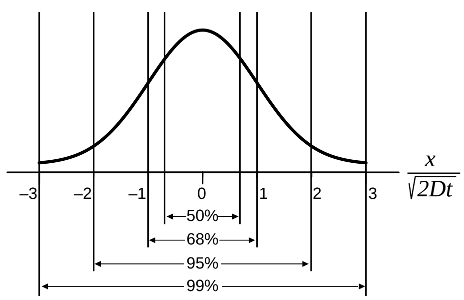


Figure 3.14

How long till half the solute diffuses to $|x|>x_{1/2}$



$$c(x,t) = \frac{n_o}{\sqrt{4\pi Dt}}\, e^{-x^2/(4Dt)}$$

$$\frac{x_{1/2}}{\sqrt{2Dt}} \lesssim \frac{2}{3}$$

$$\frac{2}{3}\sqrt{2Dt} \gtrsim x_{1/2}$$

$$\frac{4}{9}\ 2Dt \gtrsim x_{1/2}^2$$

$$t \gtrsim \frac{x_{1/2}^2}{D} \equiv t_{1/2}$$

Importance of Scale

$$t_{1/2} = \frac{x_{1/2}^2}{D} \quad ; \quad D = 10^{-5}\,\frac{\text{cm}^2}{\text{s}} \quad \text{for small solutes (e.g., Na}^+\text{)}$$

	$x_{1/2}$	$t_{1/2}$
membrane sized	10 nm	$\frac{1}{10}$ μsec
cell sized	10 μm	$\frac{1}{10}$ sec
dime sized	10 mm	10^5 sec ≈ 1 day