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Quantitative Physiology: Cells and Tissues

Homework Assignment #1

Issued: September 6, 2006

Due: September 14, 2006

Announcements

Homework will consist of *Exercises* and *Problems*. Exercises are generally more conceptual and require less number crunching. Exercises often require writing sentences to explain what you have learned. **Written solutions should be submitted for both Exercises and Problems.**

Exercise 1. The fundamental premise of the “Random Walk Model” of diffusion is that particles of solute are in constant **random** motion. However, Fick’s first law says that solute flux is down its concentration gradient — suggesting that the motion of particles is not random. Rather, particles seem to prefer to move in one direction (down the concentration gradient) more than in any other. Resolve this apparent contradiction by writing a few well-chosen sentences.

Exercise 2. Measurements show that the diffusivity of potassium ions in aqueous solutions of KCl changes very little with the concentration of KCL. The diffusivities in 10, 100, and 1000 mmol/L solutions are 1.917×10^{-5} , 1.844×10^{-5} , and 1.892×10^{-5} cm²/s. Is this consistent or inconsistent with the Random Walk Model of diffusion. Explain with a few well-chosen sentences.

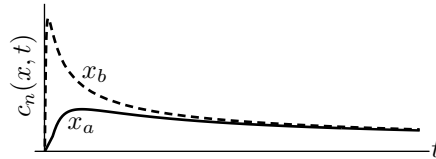
Exercise 3. Concentration is treated as a continuous quantity in Fick’s first law, in the continuity equation, and in the diffusion equation. However, matter is composed of atoms that are discrete. Therefore, concentration is actually discrete.

- Estimate the number of potassium ions in a typical red blood cell (RBC). Assume that the volume of an RBC is 90 fL. Also assume that the concentration of potassium in an RBC is 150 mmol/L. Briefly explain the physical significance of your result.
- Estimate the distance between potassium ions in a solution whose concentration of potassium is 150 mmol/L. Briefly explain the physical significance of your result.

Exercise 4. The time course of one-dimensional diffusion of a solute from a point source in space and time has the form

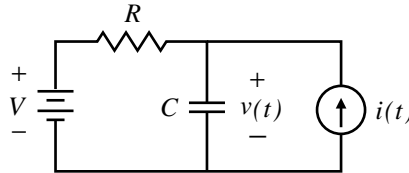
$$c_n(x, t) = \frac{n_o}{\sqrt{4\pi Dt}} e^{-x^2/4Dt},$$

where n_o is the number of moles of solute per unit area placed at $x = 0$ at $t = 0$. $c_n(x, t)$ is computed as a function of time for locations x_a and x_b , and shown in the following figure.



Is $x_a > x_b$ or is $x_a < x_b$? Explain.

Problem 1. The electric network shown below consists of a current source, a resistor, a battery (constant voltage source), and a capacitor.



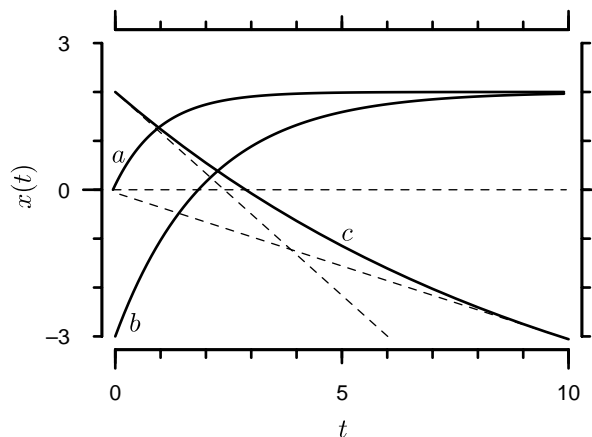
- Determine a differential equation that relates the voltage $v(t)$ to the current $i(t)$.
- For $i(t) = 0$ and $v(0) = -V$, determine $v(t)$ for $t \geq 0$. Sketch $v(t)$ versus t .
- Assume that $i(t) = 0$ for $t < 0$ and that the circuit is in steady state at $t = 0^-$. The current $i(t) = I$ for $t > 0$. Determine and sketch $v(t)$ versus t .

Problem 2. Consider an exponential function of time,

$$x(t) = Ae^{-t/\tau} + B.$$

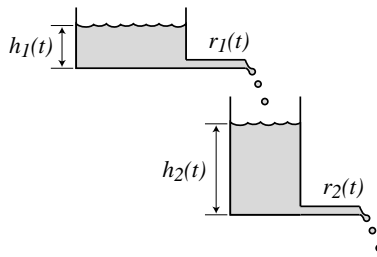
Part a. We can approximate this function at $t = 0$ by a straight line that passes through $x(0)$ with a slope equal to the slope of the exponential function. We can also approximate this function as $t \rightarrow \infty$ by a straight line through $x(\infty)$ with zero slope. Determine the time T where these two straight lines intersect.

Part b. Determine the time constants τ_a and τ_b of the exponential functions shown by the solid curves in the following plot labelled “a” and “b,” respectively. Briefly explain your method.



Part c. Determine the time constant τ_c of the exponential function shown by the solid curve in the previous plot labelled “c.” Briefly explain your method.

Problem 3. The following figure illustrates a cascaded system of two water tanks. Water flows out of the first tank and into the second at a rate $r_1(t)$, and out of the second tank at a rate $r_2(t)$.



The rates of flow out of the tanks are proportional to the heights of the water in the tanks: $r_1(t) = k_1 h_1(t)$ and $r_2(t) = k_2 h_2(t)$, where k_1 and k_2 are each $0.02 \text{ m}^2/\text{minute}$. The height of tank 1 is 1 m and that of tank 2 is 2 m. The cross-sectional area of tank 1 is $A_1 = 4 \text{ m}^2$ and that of the second tank is $A_2 = 2 \text{ m}^2$. At time $t = 0$, tank 1 is full and tank 2 is empty.

- If the height of water in tank 2 ever exceeds the height of the tank (2 m), the water will overflow. Will the water ever overflow? Explain.
- Set up a system of differential equations to determine $h_2(t)$. Solve the equations to determine an expression for $h_2(t)$.
- At what time does the water in tank 2 reach its peak? What will be the maximum height of water ever achieved in tank 2?
- At what time will the water stop flowing out of tank 1? Explain your answer in mathematical terms and then in physical terms.
- If both tanks were full at $t = 0$, would the second tank ever overflow? Explain.

Problem 4. To wiggle your big toe, neural messages travel along a single neuron that stretches from the base of your spine to your toe. Assume that the membrane of this neuron can be represented as a uniform cylindrical shell that encloses the intracellular environment, which is represented as a simple saline solution. The diameter of the shell is $10 \text{ }\mu\text{m}$ and the length is 1 m. Assume that 10^{-15} moles of dye are injected into the neuron at time $t = 0$ and at a point located in the center of the neuron, which we will refer to as the point $z = 0$. Assume that the dye diffuses across the radial dimension so quickly that the concentration of dye $c(z, t)$ depends only on the longitudinal direction z and time t . Assume that the diffusivity of the dye in the intracellular saline is $D = 10^{-7} \text{ cm}^2/\text{s}$ and that the membrane is impermeant to the dye.

Part a. Determine the amount of time t_1 required for 5% the injected dye to diffuse to points outside the region $-1 \text{ mm} < z < 1 \text{ mm}$.

Part b. Determine the amount of time t_2 required for half the injected dye to diffuse to points outside the region $-1 \text{ mm} < z < 1 \text{ mm}$. Determine the ratio of t_2 to t_1 . Briefly explain the physical significance of this result.

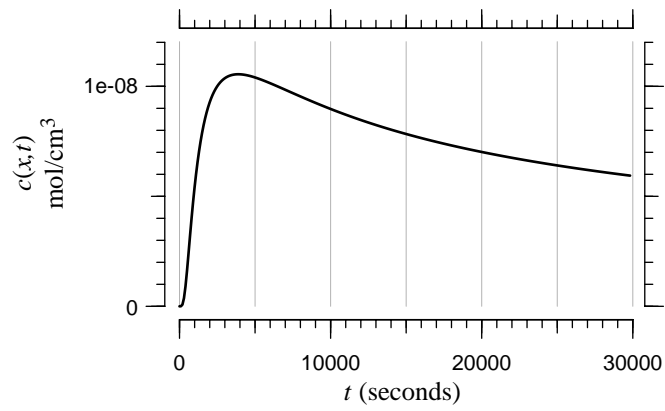
Part c. Determine the amount of time t_3 required for 5% the injected dye to diffuse to points outside the region $-10 \text{ mm} < z < 10 \text{ mm}$. Determine the ratio of t_3 to t_1 . Briefly explain the physical significance of this result.

Part d. The solution to this one-dimensional diffusion problem can be written as

$$c(z, t) = \frac{n_0}{\sqrt{4\pi Dt}} e^{-z^2/(4Dt)}$$

for distances z that are small compared to the length of the nerve and for times t that are small compared to the time it takes for the dye to diffuse to the end of the nerve. Determine the numerical value of n_0 for the dye problem described at the beginning of this problem along with its units.

Part e. The following plot shows the concentration of dye as a function of time for a particular point at $z_0 > 0$.



Determine z_0 .